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On the virtual-mass and damping coefficients for long waves in water of finite depth

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A half-immersed circular cylinder is making vertical oscillations on water of finite constant depth. The virtual-mass and damping coefficients are studied in the limit as the wavelength tends to infinity. It is found that the virtual-mass coefficient tends to a finite limit and that the amplitude ratio ultimately varies as the frequency. This behaviour differs from the behaviour for infinite depth, where the virtual-mass coefficient tends to infinity and the amplitude ratio ultimately varies as $(\text{frequency})^2$.

1. Introduction

In ship hydrodynamics it is customary to describe the hydrodynamic force on a harmonically oscillating body by means of two dimensionless coefficients. The force component in phase with the acceleration of the body is described by the virtual-mass coefficient, the force component in quadrature with the acceleration by the amplitude ratio. In the present note we shall be mainly concerned with the virtual-mass coefficient. For a half-immersed heaving circular cylinder on water of infinite depth it has been shown (Ursell 1949) that the virtual-mass coefficient tends to infinity as the frequency tends to zero, and for water of finite depth the same conclusion was suggested by the numerical computations of Yu & Ursell (1961; this paper will be referred to as Y). It has however been known for some time that the virtual-mass coefficients given in Y contain errors; thus Rhodes-Robinson (1970) found in his analytical treatment of the short-wave limit that at high frequencies his results were in agreement with unpublished computations by W. R. Porter but not with those in Y. More recently several workers have obtained numerical results for low frequencies which differ from the results in Y and also from each other, and which suggest that the virtual-mass coefficient remains finite for finite depth.

In the present paper the virtual-mass coefficient will be investigated analytically for finite depth, when the frequency tends to zero while the radius and the depth are kept fixed. As in Y, a half-immersed circular cylinder will be treated, and the potential will be expanded in terms of a complete set of harmonic functions. (For infinite depth this expansion becomes the well-known expansion in terms of a wave source and wave-free potentials.) The behaviour of these harmonic functions will be investigated in the long-wave limit, and it will be shown that one of them tends to infinity while the others tend to finite limits. (The coefficients in the expansion also tend to finite limits.) The most important step in the investigation is the determination of a certain finite limit. It will be shown that the virtual-mass coefficient tends to a finite limit, given by equation (4.8) below. A similar result may be expected for other cross-sections.

2. The form of the expansion

As in Y, let the origin of co-ordinates be taken in the mean free surface of the fluid, at the mean position of the centre of the heaving circular cylinder. Let the x axis be taken horizontal and the y axis vertical (y increasing with depth). Let a be the radius of the cylinder and h the mean water depth. Write $x = r\sin\theta$ and $y = r\cos\theta$. Then the velocity potential $\mathrm{Re}\,\phi(x,y)\,e^{-i\sigma t}$ satisfies the equation of continuity

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \quad \text{in the fluid region} \quad -\infty < x < \infty, \quad 0 < y < h, \quad r > a,$$

with the linearized boundary conditions

$$\frac{\sigma^2}{g}\phi + \frac{\partial\phi}{\partial y} \equiv K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on the free surface} \quad y = 0, \quad |x| > a, \tag{2.1}$$

$$\partial\phi/\partial y = 0 \quad \text{on the bottom} \quad y = h, \tag{2.2}$$

$$\partial\phi/\partial r = l\sigma\cos\theta \quad \text{on the cylinder} \quad r = a, \quad |\theta| < \tfrac{1}{2}\pi. \tag{2.3}$$

(Here l, which is taken to be real, is the amplitude of oscillation of the cylinder.) At infinity there is the radiation condition that waves travel outwards. The wavenumber k_0 is defined in the usual manner as the positive root of the equation $k_0 h\tanh k_0 h = Kh$; we observe that $k_0 h$ is a function of Kh, and that

$$k_0 h \sim (Kh)^{\frac{1}{2}} \quad \text{as} \quad Kh \to 0.$$

By symmetry it is sufficient to consider the region in which $x > 0$ (or equivalently, in which $0 < \theta < \frac{1}{2}\pi$). Then (as in Y) the potential is expanded in the form

$$\phi = l\sigma a\left[-A_0(F+if) + \sum_{m=1}^{\infty} a^{2m}A_{2m}(F_{2m}+if_{2m})\right], \tag{2.4}$$

where

$$F+if = -\int_0^\infty \frac{\cosh k(h-y)\cos kx\,dk}{k\sinh kh - K\cosh kh} - \frac{2\pi i\cosh k_0 h}{2k_0 h + \sinh 2k_0 h}\cosh k_0(h-y)\cos k_0 x, \tag{2.5}$$

$$\begin{aligned} F_{2n}+if_{2n} = {} & \frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}} \\ & - \frac{1}{(2n-1)!}\int_0^\infty \frac{e^{-kh}(K+k)(K\sinh ky - k\cosh ky)\,k^{2n-2}\cos kx}{k\sinh kh - K\cosh kh}\,dk \\ & + \frac{2\pi i}{(2n-1)!}\frac{k_0^{2n}\cosh k_0(h-y)\cos k_0 x}{\cosh k_0 h(2k_0 h + \sinh 2k_0 h)} \quad (n = 1,2,3,\ldots), \end{aligned} \tag{2.6}$$

where $\int$ denotes the Cauchy principal value of the integral. Each term of the expansion (2.4) satisfies the equation of continuity, the free-surface condition, the bottom condition and the radiation condition. The boundary condition (2.3) on the circle is also satisfied if the coefficients A_0 and A_{2m} can be chosen such that

$$\cos\theta = -A_0\left\langle a\frac{\partial}{\partial r}(F+if)\right\rangle + \sum_{m=1}^{\infty} a^{2m}A_{2m}\left\langle a\frac{\partial}{\partial r}(F_{2m}+if_{2m})\right\rangle, \tag{2.7}$$

when $r = a$ and $0 < \theta < \frac{1}{2}\pi$. Here and later, angular brackets are used to indicate that r is to be put equal to a. It was shown in Y that the coefficients A_0 and A_{2m} are uniquely determined by (2.7). The functions F, f, $h^{2m}F_{2m}$ and $h^{2m}f_{2m}$ are functions of Kr and θ, involving Kh (or $k_0 h$) as a dimensionless parameter; the coefficients A_0 and A_{2m} are functions of the dimensionless parameters Kh and a/h.

In the present work we are concerned with the behaviour of the potential (2.4) when $Kh \to 0$ while a/h remains fixed. For this purpose we must study the behaviour of the functions F, f, F_{2m} and f_{2m} as $Kh \to 0$ [and the consequent behaviour of the coefficients A_0 and A_{2m} determined by (2.7)]. From the expressions (2.5) and (2.6) it is evident that

$$\langle f\rangle \sim -\frac{\pi}{2k_0 h}, \quad \langle a^{2m}f_{2m}\rangle \sim \frac{\pi}{2k_0 h}\frac{(k_0 a)^{2m}}{(2m-1)!},$$

and thus $\langle f\rangle$ becomes infinite in the limit, but it is not difficult to see that $\langle a\partial f/\partial r\rangle = O(k_0 h)$ tends to zero. The functions $\langle f_{2m}\rangle$ also tend to zero. The limiting behaviour of the functions F and F_{2m} is more difficult to determine. It is shown in the appendix that they tend to finite limit functions:

$$F(x,y) \to \hat{F}(x,y) = \tfrac{1}{2}\log\left[2(\cosh \pi x/h - \cos \pi y/h)\right], \tag{2.8}$$

$$F_{2m}(x,y) \to \hat{F}_{2m}(x,y) = -\frac{1}{(2m-1)!}\left(\frac{\partial}{\partial y}\right)^{2m}\hat{F}(x,y). \tag{2.9}$$

The limit functions $\hat{F}$ and $\hat{F}_{2m}$ satisfy the boundary condition $\partial\hat{F}/\partial y = 0$ on $y = 0$ [instead of (2.1)] and also Laplace's equation and the bottom condition (2.2), together with the conditions

$$\begin{aligned} \hat{F} - \log r/h \quad &\text{is bounded near} \quad r = 0,\\ \hat{F}_{2m} - r^{-2m}\cos 2m\theta \quad &\text{is bounded near} \quad r = 0, \end{aligned}$$

but these conditions do not determine $\hat{F}$ and $\hat{F}_{2m}$ uniquely since they allow arbitrary constants to be added. It does not seem possible to determine these constants (which affect the virtual mass) except by applying a complicated limiting operation to the solution for arbitrary $k_0 h$.

3. The determination of A_0 and A_{2m} for arbitrary $k_0 h$

We can rewrite (2.7) in the form

$$\cos\theta = -A_0\left\langle a\frac{\partial F}{\partial r}\right\rangle + \sum_{m=1}^{\infty} a^{2m}A_{2m}\left\langle a\frac{\partial F_{2m}}{\partial r}\right\rangle + iE\mathfrak{A}\left\langle a\frac{\partial}{\partial r}\frac{\cosh k_0(h-y)\cos k_0 x}{(k_0 h)^2}\right\rangle, \tag{3.1}$$

where

$$\mathfrak{A} = A_0 + \frac{1}{\cosh^2 k_0 h}\sum_{m=1}^{\infty}\frac{A_{2m}(k_0 a)^{2m}}{(2m-1)!} \tag{3.2}$$

and

$$E = \frac{2\pi\cosh k_0 h}{2k_0 h + \sinh 2k_0 h}(k_0 h)^2. \tag{3.3}$$

It is not difficult to see that the last angular bracket in (3.1) tends to the finite limit $(a/h)\cos^2 2\theta - (a/h)\cos\theta$ as $k_0 h \to 0$.

Let us consider the expansions

$$\cos\theta = -B_0\left\langle a\frac{\partial F}{\partial r}\right\rangle + \sum_{m=1}^{\infty} a^{2m}B_{2m}\left\langle a\frac{\partial F_{2m}}{\partial r}\right\rangle \tag{3.4}$$

and $$\left\langle a\frac{\partial}{\partial r}\frac{\cosh k_0(h-y)\cos k_0 x}{(k_0 h)^2}\right\rangle = -C_0\left\langle a\frac{\partial F}{\partial r}\right\rangle + \sum_{m=1}^{\infty} a^{2m}C_{2m}\left\langle a\frac{\partial F_{2m}}{\partial r}\right\rangle \tag{3.5}$$

of the functions on the left in terms of the functions on the right. Evidently B_{2m} and C_{2m} are real, and

$$A_{2m} = B_{2m} - iE\mathfrak{A}C_{2m}. \tag{3.6}$$

Let us write

$$\mathfrak{B} = B_0 + \frac{1}{\cosh^2 k_0 h}\sum_{m=1}^{\infty}\frac{B_{2m}(k_0 a)^{2m}}{(2m-1)!}, \tag{3.7}$$

$$\mathfrak{C} = C_0 + \frac{1}{\cosh^2 k_0 h}\sum_{m=1}^{\infty}\frac{C_{2m}(k_0 a)^{2m}}{(2m-1)!}. \tag{3.8}$$

Then, from (3.6), we have

$$\mathfrak{A} = \mathfrak{B} - iE\mathfrak{A}\mathfrak{C},$$

whence

$$\mathfrak{A} = \frac{\mathfrak{B}}{1+iEC} = \frac{\mathfrak{B}(1-iE\mathfrak{C})}{1+E^2\mathfrak{C}^2}$$

$$= \mathfrak{B}(1-iE\mathfrak{C})/\mathfrak{D}, \tag{3.9}$$

where

$$\mathfrak{D} = 1 + E^2\mathfrak{C}^2. \tag{3.10}$$

Thus, when B_0, B_{2m}, C_0 and C_{2m} have been determined from (3.4) and (3.5), the coefficients A_0 and A_{2m} are given by (3.6), where $\mathfrak{A}$ is given by (3.9) and where $\mathfrak{B}$ and $\mathfrak{C}$ are given by (3.7) and (3.8). It is then possible to find ϕ everywhere, and in particular its value $\langle\phi\rangle$ on the circle. The real part of $\langle\phi\rangle$ is proportional to the pressure component in phase with the acceleration of the circle, and it is the vertical force derived from this component which is described by the virtual-mass coefficient. The imaginary part of $\langle\phi\rangle$ is proportional to the pressure component in quadrature with the acceleration, and is thus related to the wave amplitude at infinity.

We shall now express $\mathrm{Re}\langle\phi\rangle$ and $\mathrm{Im}\langle\phi\rangle$ in terms of B_{2m} and C_{2m}. We know from (2.4) that

$$\frac{\phi}{l\sigma a} = -A_0 F + \sum_{m=1}^{\infty} a^{2m}A_{2m}F_{2m} - i\mathfrak{A}f, \tag{3.11}$$

where

$$A_{2m} = B_{2m} - iE\mathfrak{A}C_{2m} = B_{2m} - iE\mathfrak{B}\mathfrak{D}^{-1}(1-iE\mathfrak{C})C_{2m}$$

$$= B_{2m} - \frac{\mathfrak{B}}{\mathfrak{D}}\mathfrak{C}E^2C_{2m} - \frac{iE\mathfrak{B}}{\mathfrak{D}}C_{2m} \tag{3.12}$$

and

$$\mathfrak{A} = \mathfrak{B}\mathfrak{D}^{-1}(1-iE\mathfrak{C}), \quad |\mathfrak{A}| = \mathfrak{B}/\mathfrak{D}^{\frac{1}{2}}. \tag{3.13}$$

On substituting (3.12) and (3.13) in (3.11) and taking real and imaginary parts on $r = a$, we find that

$$\frac{\mathrm{Re}\langle\phi\rangle}{l\sigma a} = \left\langle -B_0 F + \sum_1^{\infty} a^{2m}B_{2m}F_{2m}\right\rangle - \frac{\mathfrak{B}}{\mathfrak{D}}\mathfrak{C}E^2\left\langle -C_0F + \sum_1^{\infty} a^{2m}C_{2m}F_{2m}\right\rangle - \frac{\mathfrak{B}\mathfrak{C}E}{\mathfrak{D}}\langle f\rangle \tag{3.14}$$

and $$\frac{\mathrm{Im}\langle\phi\rangle}{l\sigma a} = -\frac{E\mathfrak{B}}{\mathfrak{D}}\left\langle -C_0F + \sum_1^{\infty} a^{2m}C_{2m}F_{2m}\right\rangle - \frac{\mathfrak{B}}{\mathfrak{D}}\langle f\rangle. \tag{3.15}$$

At infinity

$$\frac{\phi}{l\sigma a} = \frac{2\pi i \cosh k_0 h}{2k_0 h + \sinh 2k_0 h} \cosh k_0(h-y) \exp(ik_0|x|)\mathfrak{A} = \mathscr{F}, \quad \text{say,}$$

while the wave amplitude is $\sigma g^{-1}|\phi|$ when $y = 0$, so the amplitude ratio is

$$|\sigma\phi/gl| = \frac{\sigma^2 a}{g}|\mathscr{F}| = \frac{2\pi k_0 a \sinh k_0 h}{2k_0 h + \sinh 2k_0 h} \cosh k_0 h |\mathfrak{A}|,$$

i.e.

$$\text{amplitude ratio} = \frac{\pi \sinh 2k_0 h}{2k_0 h + \sinh 2k_0 h} k_0 a \frac{|\mathfrak{B}|}{\mathfrak{D}^{\frac{1}{2}}}, \tag{3.16}$$

from (3.13).

4. The limit of the potential for small $k_0 h$

To find this limit we consider the behaviour of (3.4) and (3.5) for small $k_0 h$. It is shown in the appendix that

$$F(x,y) = \hat{F}(x,y) + O(k_0 h)^2,$$
$$F_{2m}(x,y) = \hat{F}_{2m}(x,y) + O(k_0 h)^2,$$
$$\left\langle a\frac{\partial F}{\partial r}\right\rangle = \left\langle a\frac{\partial \hat{F}}{\partial r}\right\rangle + O(k_0 h)^2,$$
$$\left\langle a\frac{\partial F_{2m}}{\partial r}\right\rangle = \left\langle a\frac{\partial \hat{F}_{2m}}{\partial r}\right\rangle + O(k_0 h)^2,$$

where $\hat{F}$ and $\hat{F}_{2m}$ are defined by (2.8) and (2.9). It thus seems reasonable to suppose (and it can be rigorously proved) that $B_{2m} = \hat{B}_{2m} + O(k_0 h)^2$ and $C_{2m} = \hat{C}_{2m} + O(k_0 h)^2$, where

$$\cos\theta = -\hat{B}_0\left\langle a\frac{\partial \hat{F}}{\partial r}\right\rangle + \sum_{m=1}^{\infty} a^{2m}\hat{B}_{2m}\left\langle a\frac{\partial \hat{F}_{2m}}{\partial r}\right\rangle \tag{4.1}$$

and

$$\left(\frac{a}{h}\right)^2 \cos 2\theta - \left(\frac{a}{h}\right)\cos\theta = -\hat{C}_0\left\langle a\frac{\partial \hat{F}}{\partial r}\right\rangle + \sum_{m=1}^{\infty} a^{2m}\hat{C}_{2m}\left\langle a\frac{\partial \hat{F}_{2m}}{\partial r}\right\rangle. \tag{4.2}$$

[It is not difficult to show that the left-hand side of (4.2) is the limit of the left-hand side of (3.5).] These equations no longer involve $k_0 h$. It also follows that $\mathfrak{B} = \hat{B}_0 + O(k_0 h)^2$, $\mathfrak{C} = \hat{C}_0 + O(k_0 h)^2$ and $\mathfrak{D} = 1 + O(k_0 h)^2$. Thus, neglecting $(k_0 h)^2$ in (3.14), we find that

$$\frac{\operatorname{Re}\langle\phi\rangle}{l\sigma a} = \left\langle -\hat{B}_0\hat{F} + \sum_{m=1}^{\infty} a^{2m}\hat{B}_{2m}\hat{F}_{2m}\right\rangle + \tfrac{1}{4}\pi^2\hat{B}_0\hat{C}_0 + O(k_0 h)^2 \tag{4.3}$$
$$= \langle\hat{B}(x,y)\rangle + \tfrac{1}{4}\pi^2\hat{B}_0\hat{C}_0 + O(k_0 h)^2,$$

where

$$\hat{B}(x,y) = -\hat{B}_0\hat{F} + \sum_{m=1}^{\infty} a^{2m}\hat{B}_{2m}\hat{F}_{2m}. \tag{4.4}$$

The constants $\hat{B}_0$ and $\hat{C}_0$ can be found explicitly, for the expansions of $\langle a\partial\hat{F}/\partial r\rangle$ and $\langle a\partial\hat{F}_{2m}/\partial r\rangle$ in Fourier series (obtained from equations (A 17) and (A 19) in the appendix) show that

$$\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial \hat{F}}{\partial r}\right\rangle d\theta = \tfrac{1}{2}\pi, \quad \int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial \hat{F}_{2m}}{\partial r}\right\rangle d\theta = 0.$$

Thus, on applying the operator $\int_0^{\frac{1}{2}\pi} \ldots d\theta$ to (4.1) we find that $1 = -\frac{1}{2}\pi\hat{B}_0$, i.e. that $\hat{B}_0 = -2/\pi$. Similarly, it is found from (4.2) that $\hat{C}_0 = 2a/\pi h$; thus

$$\mathrm{Re}\,\langle\phi\rangle/l\sigma a = \langle\hat{B}(x,y)\rangle - a/h + O(k_0 h)^2. \tag{4.5}$$

Also, from (3.15),

$$\begin{aligned}\mathrm{Im}\,\langle\phi\rangle/l\sigma a &= -\hat{B}_0 f(1+O(k_0h)^2)\\ &= -(k_0h)^{-1}(1+O(k_0h)^2),\end{aligned} \tag{4.6}$$

which tends to infinity as $k_0 h \to 0$. And from (3.16) the amplitude ratio is seen to be

$$\tfrac{1}{2}\pi(k_0 a)\,|\hat{B}_0|(1+O(k_0h)^2) = (k_0a)(1+O(k_0h)^2). \tag{4.7}$$

The virtual-mass coefficient is defined as the ratio

$$\begin{aligned}&-\sigma\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{Re}\,\langle\phi\rangle\, a\cos\theta\, d\theta/\tfrac{1}{2}\pi a^2 l\sigma^2\\ &\qquad = -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left(\hat{B}(a\sin\theta, a\cos\theta) - \frac{a}{h}\right)\cos\theta\, d\theta + O(k_0h)^2.\end{aligned} \tag{4.8}$$

These results may be compared with the well-known corresponding results for infinite depth when $Ka \to 0$ (with the corresponding potential denoted by ϕ_∞):

$$\frac{\mathrm{Re}\,\langle\phi_\infty\rangle}{l\sigma a} = \frac{2}{\pi}\log\frac{1}{Ka} + O(1), \quad \frac{\mathrm{Im}\,\langle\phi_\infty\rangle}{l\sigma a} = 2 + o(1),$$

$$\text{amplitude ratio} \sim 2Ka,$$

$$\text{virtual-mass coefficient} = \frac{8}{\pi^2}\log\frac{1}{Ka} + O(1).$$

The additive constant in the potential may also be found by an application of Green's theorem to the harmonic functions ϕ and $G = F + if$ in the region occupied by the fluid. We have

$$\int\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right)ds = 0,$$

where the integral is taken along the complete boundary closed at infinity. The boundary conditions satisfied by ϕ and G show that the integrand vanishes along the free surface and the bottom, also there is no contribution from infinity, where both ϕ and G satisfy radiation conditions. Thus only the contribution from the circle $r = a$ remains, i.e.

$$\int_0^{\frac{1}{2}\pi} a\left\langle\phi\frac{\partial G}{\partial r}\right\rangle d\theta - \int_0^{\frac{1}{2}\pi} a\left\langle G\frac{\partial\phi}{\partial r}\right\rangle d\theta = 0. \tag{4.9}$$

On the circle $r = a$ we have

$$\langle G\rangle = \langle F\rangle + i\langle f\rangle = \langle\hat{F}\rangle - i\pi/2k_0h + O(k_0h), \tag{4.10}$$

$$\left\langle a\frac{\partial G}{\partial r}\right\rangle = \left\langle a\frac{\partial\hat{F}}{\partial r}\right\rangle - \tfrac{1}{2}i\pi k_0h\left\{\left(\frac{a}{h}\right)^2\cos 2\theta - \frac{a}{h}\cos\theta\right\} + O(k_0h)^2, \tag{4.11}$$

$$\langle\partial\phi/\partial r\rangle = l\sigma a\langle\partial\hat{B}/\partial r\rangle = l\sigma\cos\theta; \tag{4.12}$$

these are known results from the expression for F and from the boundary condition. The form of $\langle\phi\rangle$ is given by (4.3) and (4.4):

$$\langle\phi\rangle = l\sigma a\langle\hat{B}(x,y)\rangle + c(k_0h) + O(k_0h), \tag{4.13}$$

where $c(k_0h)$ is independent of θ. In fact we see that

$$c(k_0h) = (k_0h)^{-1}p + q + O(k_0h), \tag{4.14}$$

where p and q are constants. Alternatively it would be reasonable to *assume* the forms (4.13) and (4.14) since substitution in (4.9) shows that $c(k_0h)$ must increase at least as fast as $(k_0h)^{-1}$. The constants p and q can be found by substituting (4.10)–(4.13) in (4.9). We thus obtain

$$0 = \frac{1}{k_0h}\left\{p\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial\hat{F}}{\partial r}\right\rangle d\theta + \tfrac{1}{2}i\pi\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial\hat{B}}{\partial r}\right\rangle d\theta\right\} + \int_0^{\frac{1}{2}\pi}\left\langle\hat{B}\frac{\partial\hat{F}}{\partial r} - \hat{F}\frac{\partial\hat{B}}{\partial r}\right\rangle a\,d\theta \tag{4.15}$$

$$+q\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial\hat{F}}{\partial r}\right\rangle d\theta - \tfrac{1}{2}i\pi p\int_0^{\frac{1}{2}\pi}\left\{\left(\frac{a}{h}\right)^2\cos 2\theta - \frac{a}{h}\cos\theta\right\}d\theta + O(k_0h), \tag{4.16}$$

where all the integrals are independent of k_0h. We now observe that

$$\int_0^{\frac{1}{2}\pi}\left\langle\hat{B}\frac{\partial\hat{F}}{\partial r} - \hat{F}\frac{\partial\hat{B}}{\partial r}\right\rangle a\,d\theta = 0, \tag{4.17}$$

and that

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial\hat{F}}{\partial r}\right\rangle d\theta = \int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial\hat{B}}{\partial r}\right\rangle d\theta = 1. \tag{4.18a, b}$$

Let us, for instance, consider (4.17). By Green's theorem we have

$$\int\left(\hat{B}\frac{\partial\hat{F}}{\partial n} - \hat{F}\frac{\partial\hat{B}}{\partial n}\right)ds = 0,$$

where the integral is taken along the boundary of the fluid. The integrand vanishes along the free surface and the bottom, where $\partial\hat{F}/\partial y = 0$ and $\partial\hat{B}/\partial y = 0$, and also at infinity since $\hat{B}(x,y)$ has the form (4.4) and the terms involving $\hat{F}_{2m}$ vanish at infinity. (It is at this point that we use the normalization of $\hat{B}(x,y)$ by the correct additive constant.) The result (4.17) follows. Equation (4.18a) expresses the source strength of $\hat{F}$, while (4.18b) follows from the boundary condition satisfied by $\hat{B}(x,y)$ on the circle $r = a$. We can now conclude that (4.15) and (4.16) both vanish, whence it follows that

$$p = -i, \quad q = -a/h.$$

Thus $\mathrm{Re}\langle\phi\rangle/l\sigma a = \langle\hat{B}(x,y)\rangle - a/h + O(k_0h)$, in agreement with (4.5), which shows that the last term is in fact $O(k_0h)^2$.

5. Discussion

We observe that the virtual-mass coefficient involves the harmonic function $\hat{B}(x,y;a/h)$, which satisfies

$$a\,\partial\hat{B}/\partial r = \cos\theta \quad \text{on} \quad r = a \tag{5.1}$$

and

$$\partial\hat{B}/\partial y = 0 \quad \text{on} \quad y = h, \tag{5.2}$$

but satisfies the limiting boundary condition

$$\partial\hat{B}/\partial y = 0 \quad \text{on the free surface} \tag{5.3}$$

instead of the usual free-surface condition (2.1). $\hat{B}(x,y)$ is just one of the solutions of the limiting boundary-value problem defined by (5.1)–(5.3); evidently there are infinitely many other solutions (with finite velocity at infinity), which differ from $\hat{B}(x,y)$ by arbitrary additive constants. The theoretical calculation given above shows that the particular solution $\hat{B}(x,y)$ which we require is uniquely defined as having an expansion of the form (4.4). Since

$$\hat{F}(x,y)-\pi|x|/2h\to 0 \quad \text{and} \quad \hat{F}_{2m}\to 0$$

as $|x|\to\infty$ we may describe $\hat{B}(x,y)$ as being that solution of the limiting boundary-value problem which satisfies $\hat{B}(x,y)-|x|/h\to 0$ as $|x|\to\infty$.

We also observe that for both finite and infinite depth the quotient $\langle\phi\rangle/l\sigma a$ tends to infinity when the wavelength tends to infinity, and that for finite depth it is ultimately in quadrature with the acceleration of the cylinder whereas for infinite depth it is ultimately in phase. It is now not difficult to see how errors may arise in the numerical calculation of $\mathrm{Re}\langle\phi\rangle$ for finite depth. Let us suppose, for instance, that we solve the problem in the most direct way, by determining the complex-valued constants A_{2m} from (2.7). Then $\phi/l\sigma a$ is given by (3.11), and the difficulty arises from the last term $-i\mathfrak{A}f$ on the right-hand side, which contributes $\langle f\rangle\,\mathrm{Im}\,\mathfrak{A}$ to the quantity $\mathrm{Re}\langle\phi\rangle/l\sigma a$ appearing in the virtual-mass coefficient. Since $\langle f\rangle$ tends to infinity when the wavelength tends to infinity, we see that a small error in $\mathrm{Im}\,\mathfrak{A}$ may cause a significant error in the virtual-mass coefficient. In our method, on the other hand, the quantity $\mathrm{Re}\langle\phi\rangle/l\sigma a$ is given by (3.14), and all the terms in this expression remain bounded when the wavelength tends to infinity. In particular, the quantity $\langle f\rangle$ now occurs in the combination $E\langle f\rangle$, which remains bounded.

The present calculation uses an expansion in terms of multipole potentials at the origin, and is concerned with the half-immersed circular cylinder. Similar arguments may be expected to apply to integral equations and therefore to arbitrary cross-sections.

Appendix. Long-wave limits of source and multipole potentials

The real part of the *source potential* is, by definition [Y, equations (2.8)–(2.13)],

$$\begin{aligned}F(x,y) &= \fint_0^\infty \frac{\cosh k(h-y)\cos kx\,dk}{K\cosh kh-k\sinh kh}\\ &= \fint_0^\infty e^{-ky}\frac{\cos kx}{K-k}\,dk+\fint_0^\infty \frac{e^{-kh}(K\sinh ky-k\cosh ky)\cos kx\,dk}{(K-k)(K\cosh kh-k\sinh kh)}\\ &= (\gamma+\log Kr)\sum_{s=0}^{\infty}\frac{(-Kr)^s}{s!}\cos s\theta \qquad \text{(A 1)}\end{aligned}$$

$$\left.\begin{aligned}-\theta\sum_{s=1}^{\infty}\frac{(-Kr)^s}{s!}\sin s\theta-\sum_{s=1}^{\infty}\frac{(-Kr)^s}{s!}\left(1+\tfrac{1}{2}+\dots+\frac{1}{s}\right)\cos s\theta\\ +Kh\sum_{s=0}^{\infty}\frac{1}{(2s+1)!}G_{2s+1}(Kh)\left(\frac{r}{h}\right)^{2s+1}\cos(2s+1)\theta\end{aligned}\right\} \qquad \text{(A 2)}$$

$$-\sum_{s=0}^{\infty}\frac{1}{(2s)!}G_{2s+1}(Kh)\left(\frac{r}{h}\right)^{2s}\cos 2s\theta, \qquad \text{(A 3)}$$

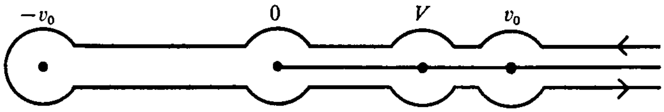

FIGURE 1. The contour of integration C.

where, by definition,

$$G_{2s+1}(V) = \int_0^\infty \frac{u^{2s+1}e^{-u}\,du}{(V-u)(V\cosh u - u\sinh u)}. \tag{A 4}$$

We shall examine the behaviour of $G_{2s+1}(V)$ when $V \to 0$, and shall then be able to infer the behaviour of $F(x, y)$ when $Kh \to 0$. Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{u^{2s+1}e^{-u}(\log u - \pi i)\,du}{(u-V)(u\sinh u - V\cosh u)} \tag{A 5}$$

along a contour C which will be specified in a moment. We observe that the integrand in (A 5) has a logarithmic branch point at $u = 0$, and that near $u = 0$ it has poles at $u = V$ and at $u = \pm v_0$, where v_0 is the positive root of $v_0 \tanh v_0 = V$. Let the contour of integration C be taken as in figure 1. A cut is made from $u = 0$ to $u = +\infty$ along the real u axis. For positive u we easily see that $\log u - \pi i$ is equal to $\log|u| - \pi i$ along the upper side of the cut, but equal to $\log|u| + \pi i$ along the lower side of the cut; for negative u the value of $\log u - \pi i$ is $\log|u|$ on both the upper and the lower part of the contour. It is then not difficult to see that

$$\begin{aligned}\frac{1}{2\pi i}\int_C &= \int_0^\infty \frac{u^{2s+1}e^{-u}\,du}{(u-V)(u\sinh u - V\cosh u)} \\ &+ \tfrac{1}{2}(\text{residue at } u = v_0) + \tfrac{1}{2}(\text{residue at } u = V) + (\text{residue at } u = v_0 e^{\pi i}) \\ &+ \tfrac{1}{2}(\text{residue at } u = v_0 e^{2\pi i}) + \tfrac{1}{2}(\text{residue at } u = Ve^{2\pi i}),\end{aligned} \tag{A 6}$$

where the residue terms come from the indentations at the poles. It is found that the residue terms from $u = v_0$, $v_0 e^{\pi i}$ and $v_0 e^{2\pi i}$ add up to zero, and that the residue terms from $u = V$ and $Ve^{2\pi i}$ add up to $-V^{2s}\log V$. Thus

$$\frac{1}{2\pi i}\int_C = \int_0^\infty \frac{u^{2s+1}e^{-u}\,du}{(u-V)(u\sinh u - V\cosh u)} - V^{2s}\log V,$$

i.e.

$$G_{2s+1}(V) = \frac{1}{2\pi i}\int_C + V^{2s}\log V. \tag{A 7}$$

We now show, by deforming the contour C, that

$$\frac{1}{2\pi i}\int_C$$

has a series expansion in powers of V. Let the contour C be deformed into the contour C_1 shown in figure 2, where $\eta < \frac{1}{2}\pi$ is some fixed positive constant. By Cauchy's theorem, $\int_C = \int_{C_1}$. It is not difficult to see that $|u \tanh u|$ and $|u|$ have positive lower bounds on C_1, thus there is a number $m(\eta) > 0$ such that

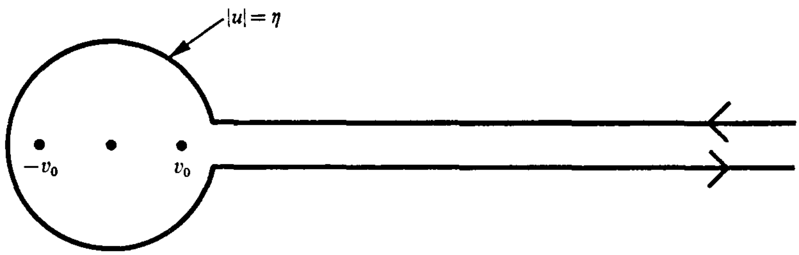

FIGURE 2. The contour C_1.

$|u \tanh u| > m(\eta)$ and $|u| > m(\eta)$. Choose V such that $|V| < m(\eta)$; then the integrand can be expanded in powers of V:

$$\frac{u\,e^{-u}}{(u-V)\,(u\sinh u - V\cosh u)} = \sum_{m=0}^{\infty} \frac{V^m}{u^{m+1}} (\coth^{m+1} u - 1).$$

It follows that

$$\frac{1}{2\pi i}\int_{C_1} = \sum_{m=0}^{\infty} V^m I(2s-m-1, m+1),$$

where by definition

$$I(M,N) = \frac{1}{2\pi i}\int_{\infty}^{0+} u^M (\log u - \pi i)\,(\coth^N u - 1)\,du. \tag{A 8}$$

(The contour of integration in (A 8) is C_1 or any equivalent contour.) Thus, finally,

$$G_{2s+1}(V) = \frac{1}{2\pi i}\int_{C_1} + V^{2s}\log V,$$

i.e.

$$G_{2s+1}(V) = V^{2s}\log V + \sum_{m=0}^{\infty} V^m I(2s-m-1, m+1). \tag{A 9}$$

This is the required expansion for $G_{2s+1}(V)$. Let this be substituted in (A 1)–(A 3). The terms (A 2) and (A 3) are of the form

$$Kh \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} (Kh)^{2s} \log Kh \left(\frac{r}{h}\right)^{2s+1} \cos (2s+1)\,\theta$$
$$- \sum_{s=0}^{\infty} \frac{1}{(2s)!} (Kh)^{2s} \log Kh \left(\frac{r}{h}\right)^{2s} \cos 2s\theta$$
$$- \sum_{s=0}^{\infty} \frac{1}{(2s)!} I(2s-1, 1) \left(\frac{r}{h}\right)^{2s} \cos 2s\theta + O(Kh)$$
$$= -\log Kh \sum_{s=0}^{\infty} \frac{(-Kr)^s}{s!} \cos s\theta, \tag{A 10}$$
$$- \sum_{s=0}^{\infty} \frac{1}{(2s)!} I(2s-1, 1) \left(\frac{r}{h}\right)^{2s} \cos 2s\theta + O(Kh). \tag{A 11}$$

When the term (A 10) is combined with (A 1), all terms involving $\log K$ disappear, and we find that

$$(\text{A } 1) + (\text{A } 10) = \left(\gamma + \log\frac{r}{h}\right) \sum_{s=0}^{\infty} \frac{(-Kr)^s}{s!} \cos s\theta$$
$$= \gamma + \log r/h + O(Kh).$$

Thus
$$F(x,y) = \hat{F}(x,y) + O(Kh),$$
where
$$\hat{F}(x,y) = \gamma + \log\frac{r}{h} - \sum_{s=0}^{\infty}\frac{1}{(2s)!}I(2s-1,1)\left(\frac{r}{h}\right)^{2s}\cos 2s\theta.$$

It only remains to find the coefficients
$$I(2s-1,1) = \frac{1}{2\pi i}\int_{\infty}^{0+} u^{2s-1}(\log u - \pi i)(\coth u - 1)\,du$$
$$= \frac{1}{2\pi i}\int_{\infty}^{0+}\frac{u^{2s-1}e^{-u}}{\sinh u}(\log u - \pi i)\,du.$$

For this purpose, let us take ν to be real and strictly positive, and consider the integral
$$\mathscr{J}(\nu) = \frac{1}{2\pi i}\int_{\infty}^{0+}\frac{(u e^{-\pi i})^{\nu} e^{-u}}{\sinh u}\,du = \frac{1}{2^{\nu+1}\pi i}\int_{\infty}^{0+}\frac{(w e^{-\pi i})^{\nu}\,dw}{e^{w}-1}, \tag{A 12}$$
where $w = 2u$. From Erdélyi (1953, p. 32, equation (9)) we see that
$$\mathscr{J}(\nu) = \frac{\sin\pi\nu}{\pi}\frac{\Gamma(\nu+1)}{2^{\nu}}\zeta(\nu+1) \tag{A 13}$$
$$= -\frac{1}{\Gamma(-\nu)}\frac{\zeta(\nu+1)}{2^{\nu}}, \tag{A 14}$$
where
$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^{s}}$$
is the Riemann zeta-function. By analytic continuation these relations hold throughout the complex ν plane. By differentiating with respect to ν we find that
$$\mathscr{J}'(\nu) = \frac{1}{2\pi i}\int_{\infty}^{0+}\frac{(u e^{-\pi i})^{\nu}(\log u - \pi i)\,e^{-u}du}{\sinh u},$$
whence
$$I(2s-1,1) = -\mathscr{J}'(2s-1)$$
$$= -\frac{\partial}{\partial\nu}\left[\frac{\sin\pi\nu}{\pi}\frac{\Gamma(\nu+1)}{2^{\nu}}\zeta(\nu+1)\right]_{\nu=2s-1} \tag{A 15}$$
$$= +\frac{\partial}{\partial\nu}\left[\frac{1}{\Gamma(-\nu)}\frac{1}{2^{\nu}}\zeta(\nu+1)\right]_{\nu=2s-1}. \tag{A 16}$$

When $s = 1,2,3,\ldots$, we use (A 15) and find that
$$I(2s-1,1) = \frac{\Gamma(2s)}{2^{2s-1}}\zeta(2s) = \frac{(2s-1)!}{2^{2s-1}}\zeta(2s) \quad \text{when} \quad s = 1,2,3,\ldots.$$

When $s = 0$, we use (A 16) and find that
$$-I(-1,1) = -2\Gamma'(1)\zeta(0) + 2\zeta(0)\log 2 - 2\zeta'(0)$$
$$= -\gamma - \log 2 + \log 2\pi$$
$$= \log\pi - \gamma,$$
since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$ (Erdélyi 1953, p. 34, equation (18)). It follows from (A 11) that
$$F(x,y) = \hat{F}(x,y) + O(Kh),$$
where
$$\hat{F}(x,y) = \log\frac{\pi r}{h} - \sum_{s=1}^{\infty}\frac{1}{s}\left(\frac{r}{2h}\right)^{2s}\zeta(2s)\cos 2s\theta. \tag{A 17}$$

The series in (A 17) can be summed explicitly since the right side of (A 17) is the real part of

$$\log\frac{\pi\eta}{h}+\sum_{s=1}^{\infty}\log\left\{1-\left(\frac{\eta}{2sh}\right)^2\right\}$$
$$=\log\left(2\sin\frac{\pi\eta}{2h}\right)=\log\left(2\sin\frac{\pi(y+ix)}{2h}\right),\quad\text{where}\quad\eta=re^{i\theta}.$$

Thus $$\hat{F}(x,y)=\log\left|2\sin\frac{\pi(y+ix)}{2h}\right|=\tfrac{1}{2}\log\left|4\sin\frac{\pi(y+ix)}{2h}\sin\frac{\pi(y-ix)}{2h}\right|,$$

i.e. $$\hat{F}(x,y)=\tfrac{1}{2}\log\left\{2\left(\cosh\frac{\pi x}{h}-\cos\frac{\pi y}{h}\right)\right\}.$$

This is the limit of the source potential, with the correct additive constant.

To find the corresponding long-wave behaviour of the real part $F_{2n}(x,y)$ of the multipole potential, we may use the expansion

$$F_{2n}(x,y)=\frac{\cos 2n\theta}{r^{2n}}+\frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}}$$
$$+\frac{Kh}{(2n-1)!\,h^{2n}}\sum_{s=0}^{\infty}\frac{1}{(2s+1)!}\mathscr{I}_{2n+2s-1}(Kh)\left(\frac{r}{h}\right)^{2s+1}\cos(2s+1)\theta$$
$$-\frac{1}{(2n-1)!\,h^{2n}}\sum_{s=0}^{\infty}\frac{1}{(2s)!}\mathscr{I}_{2n+2s-1}(Kh)\left(\frac{r}{h}\right)^{2s}\cos 2s\theta,$$

where $$\mathscr{I}_{2s+1}(V)=\int_0^\infty\frac{(u+V)\,u^{2s+1}e^{-u}du}{V\cosh u-u\sinh u}$$
$$=V^2G_{2s+1}(V)-G_{2s+3}(V).\tag{A 18}$$

(It follows from (A 9) that $\mathscr{I}_{2s+1}(V)$ can be expanded in a power series in V.) Or we may observe more simply that

$$F_{2n}(x,y)=-\frac{1}{(2n-1)!}\left(\frac{\partial^2}{\partial y^2}-K^2\right)\left(\frac{\partial}{\partial y}\right)^{2n-2}F(x,y),$$

whence it follows that

$$F_{2n}(x,y)\to\hat{F}_{2n}(x,y)=-\frac{1}{(2n-1)!}\left(\frac{\partial}{\partial y}\right)^{2n}\hat{F}(x,y),$$

i.e.

$$\hat{F}_{2n}(x,y)=\frac{\cos 2n\theta}{r^{2n}}+\frac{2}{(2n-1)!}\frac{1}{(2h)^{2n}}\sum_{s=0}^{\infty}\frac{(2n+2s-1)!}{(2s)!}\left(\frac{r}{2h}\right)^{2s}\zeta(2n+2s)\cos 2s\theta.\tag{A 19}$$

More precisely, $F_{2n}(x,y)=\hat{F}_{2n}(x,y)+O(Kh)$.

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